

KOSZUL HYPERSURFACES OVER THE EXTERIOR ALGEBRAS

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ABSTRACT. We prove that if E is an exterior algebra over a field, h is a quadratic form, then $E/(h)$ is Koszul if and only if h is a product of two linear forms.

1. THE MAIN RESULT

Let k be a field. In this note, we consider graded k -algebras which are quotients of the polynomial or the exterior algebras with the standard grading. Let R be such a standard graded over k . We say that R is a *Koszul algebra* if the residue field k has a linear free resolution over R . For recent surveys on Koszul algebras, the reader may consult [5], [7].

While the Koszul property was studied intensively for commutative algebras, it is less well-known for quotients of exterior algebras. Any exterior algebra is Koszul because the Cartan complex is the linear free resolution for the residue field; see [2, Section 2]. The main result of this note says, quite unexpectedly, that exterior Koszul hypersurfaces are reducible. It answers in the positive an intriguing question of Phong Thieu in his thesis [10, Question 5.2.8].

Notation. If V is a k -vector space of dimension n with a fixed k -basis e_1, \dots, e_n , then the notation $k\langle e_1, \dots, e_n \rangle$ denotes the exterior algebra $\bigwedge V$. It is a graded-commutative k -algebra with the grading induced by $\deg e_i = 1$ for $1 \leq i \leq n$, so that for all homogeneous elements $a, b \in E = k\langle e_1, \dots, e_n \rangle$, we have

$$a \wedge b = (-1)^{(\deg a)(\deg b)} b \wedge a$$

and

$$a \wedge a = 0,$$

if $\deg a$ is odd. For simplicity, we will denote $a \wedge b$ simply by ab .

The main result of this note is

Theorem 1.1. *Let $E = k\langle e_1, \dots, e_n \rangle$ be an exterior algebra (where $n \geq 1$) and h a homogeneous form. Then $E/(h)$ is Koszul if and only if h is a reducible quadratic form, namely h is a product of two linear forms.*

This is in stark contrast with the fact that any commutative quadratic hypersurface is Koszul.

2. PROOF OF THE MAIN RESULT

The main work in our result is done by the following

Theorem 2.1. *Let $h = e_1 f_1 + e_2 f_2 + \dots + e_n f_n$ be a quadratic form in the exterior algebra in $2n$ variables $E = k\langle e_1, \dots, e_n, f_1, \dots, f_n \rangle$, where $n \geq 2$. Then $E/(h)$ is not Koszul.*

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Let M be a finitely generated graded R -module. If R is a quotient of an exterior algebra, this means that M is finitely generated, graded, and on M there are left and right R -actions such that

$$am = (-1)^{(\deg a)(\deg m)} ma$$

for every homogeneous elements $a \in R, m \in M$.

Denote $\beta_{i,j}^R(M) = \dim_k \operatorname{Tor}_i^R(M, k)_j$ the (i, j) -graded Betti number of M , where $i, j \in \mathbb{Z}$. The i th total Betti number of M is $\beta_i^R(M) = \dim_k \operatorname{Tor}_i^R(M, k)$. We let $t_i^R(M) = \sup\{j : \beta_{i,j}^R(M) \neq 0\}$ for each $i \in \mathbb{Z}$. The last theorem will be deduced from the following analog of [3, Main Theorem, (1)], which is about syzygies of Koszul algebras in the commutative setting.

Proposition 2.2. *Let $Q \rightarrow R$ be a surjection of Koszul algebras. Let K_\bullet be the minimal free resolution of k over Q . Denote by Z_i the i -th cycle of the complex $K_\bullet \otimes_Q R$. Then $\operatorname{reg}_R Z_i \leq \operatorname{reg}_R Z_{i-1} + 2$ for all $i \geq 0$. In particular, $t_i^Q(R) \leq 2i$ for all $i \geq 0$.*

Proof. The proof is similar to that of [4, Lemma 2.10]. Denote by B_\bullet, H_\bullet the boundary and homology of the complex $C_\bullet = K_\bullet \otimes_Q R$. For each $i > 0$, we have short exact sequences

$$\begin{aligned} 0 \rightarrow Z_i \rightarrow C_i \rightarrow B_{i-1} \rightarrow 0, \\ 0 \rightarrow B_{i-1} \rightarrow Z_{i-1} \rightarrow H_{i-1} \rightarrow 0. \end{aligned}$$

Therefore combining with the fact that K is a linear resolution, we get

$$\begin{aligned} \operatorname{reg}_R Z_i &\leq \max\{\operatorname{reg}_R C_i, \operatorname{reg}_R B_{i-1} + 1\} \\ &\leq \max\{i, \operatorname{reg}_R Z_{i-1} + 1, \operatorname{reg}_R H_{i-1} + 2\}. \end{aligned}$$

Let \mathfrak{m} be the graded maximal ideal of R . Since $H_i \cong \operatorname{Tor}_i^Q(k, R)$, we have $\mathfrak{m}H_i = 0$ for all $i \geq 0$. Now $\operatorname{reg}_R k = 0$ since R is a Koszul algebra, hence $\operatorname{reg}_R H_{i-1} = t_0^R(H_{i-1}) \leq t_0^R(Z_{i-1}) \leq \operatorname{reg}_R Z_{i-1}$. Hence

$$\operatorname{reg}_R Z_i \leq \max\{i, \operatorname{reg}_R Z_{i-1} + 2\}.$$

As $Z_i \subseteq C_i$ and $t_0^R(C_i) = i$, we get $\operatorname{reg}_R Z_i \geq t_0^R(Z_i) \geq i$. Therefore $\operatorname{reg}_R Z_i \leq \operatorname{reg}_R Z_{i-1} + 2$. By induction, $\operatorname{reg}_R Z_i \leq 2i$ for all $i \geq 0$. For the second statement, we have $t_i^Q(R) = t_0^Q(H_i) = t_0^R(H_i) \leq t_0^R(Z_i) \leq \operatorname{reg}_R Z_i \leq 2i$. \square

Lemma 2.3. *Let $R = E/(h)$ where $E = k \langle e_1, \dots, e_n, f_1, \dots, f_n \rangle$ and $h = e_1 f_1 + e_2 f_2 + \dots + e_n f_n$. Then $\beta_{2,n+2}^E(R) \neq 0$.*

Proof. This follows since the identity $e_1 \cdots e_n h = 0$ gives a minimal second syzygy of degree $n + 2$ of R as an E -module; see the proof of [9, Theorem 1.2]. \square

Proof of Theorem 2.1. For $n \geq 3$, from Lemma 2.3 and Proposition 2.2, we get the result. For $n = 2$ then $E = k \langle e_1, e_2, f_1, f_2 \rangle$, $h = e_1 f_1 + e_2 f_2$. Denote $A = E/(h)$, and

$$P_k^A(t) = \sum_{i=0}^{\infty} \beta_i^R(k) t^i \in \mathbb{Z}[[t]]$$

the Poincaré series of k . If A were Koszul, we obtain an equality

$$P_k^A(t) H_A(-t) = 1,$$

where $H_A(t)$ is the Hilbert series of A . Note that $H_A(t) = 1 + 4t + 5t^2$, hence

$$1/H_A(-t) = 1 + 4t + 11t^2 + 24t^3 + 41t^4 + 44t^5 - 29t^6 - \dots$$

which cannot be the non-negative series $P_k^A(t)$. Hence A is not Koszul. \square

Example 2.4. Consider $E = \mathbb{Q}[e_1, e_2, e_3, f_1, f_2, f_3]$ and $h = e_1f_1 + e_2f_2 + e_3f_3$. The Betti table of the minimal free resolution of \mathbb{Q} over $E/(h)$ is given by Macaulay2 [8] as follow

	0	1	2	3	4	5	6	7
total:	1	6	22	76	302	1272	5189	20614
0:	1	6	22	62	148	314	610	1106
1:
2:	.	.	.	14	154	958	4383	16372
3:
4:	196	3136

Proof of Theorem 1.1. We can assume that h is a quadratic form.

The “if” part is easy and was pointed out in [10, Theorem 5.1.5]; we include an argument here. There is nothing to do if $n = 1$ or $h = 0$, so we can assume that $n \geq 2$ and $h \neq 0$. If h is reducible, by a suitable change of coordinates, we can assume that $h = e_1e_2$. Then $E/(h)$ has a Koszul filtration in the sense of [6] hence it is Koszul.

The “only if” part: if h is not reducible, by a suitable change of coordinates, we can assume that $h = e_1e_2 + e_3e_4 + \dots + e_{2i-1}e_{2i}$ for some $2 \leq i \leq n/2$. Then $k\langle e_1, \dots, e_{2i} \rangle / (h)$ is not Koszul by Theorem 2.1, and it is an algebra retract of $E/(h)$. Therefore $E/(h)$ is also not Koszul. \square

3. FINAL REMARKS

Remark 3.1. The theory of Gröbner basis was extended to the exterior algebras by Aramova, Herzog and Hibi in [2]. It is also true as in the commutative case that, if J is a homogeneous ideal in the exterior algebra E and J has quadratic Gröbner basis with respect to some term order on E , then E/J is Koszul (see [10, Theorem 5.1.5]).

Note however that the ideal $I = (e_1f_1 + e_2f_2 + \dots + e_nf_n)$ of $E = k\langle e_1, \dots, e_n, f_1, \dots, f_n \rangle$ (where $n \geq 2$) does not have a quadratic Gröbner basis in the natural coordinates $e_1, \dots, e_n, f_1, \dots, f_n$. Indeed, we can assume that e_1f_1 is the initial form of $e_1f_1 + e_2f_2 + \dots + e_nf_n$. Now clearly $e_1(e_2f_2 + \dots + e_nf_n) \in I$, and its initial form is not divisible by e_1f_1 , so any Gröbner basis of I must contain a cubic form.

Remark 3.2. (i) Note that the ideal $J = (e_1e_2 - f_1f_2, e_1f_1 - e_2f_2)$ of $E = k\langle e_1, e_2, f_1, f_2 \rangle$ is generated by two irreducible quadratic forms, but E/J is Koszul. In fact, it has a Koszul filtration; see [10, Example 5.2.2(ii)]. On the other hand, it is shown in *loc. cit.* that J has no quadratic Gröbner basis in the natural coordinates.

(ii) Note that if $E = k\langle e_1, \dots, e_n \rangle$ be an exterior algebra and I be an ideal generated by reducible quadratic forms then E/I is not Koszul in general. For example, $E = \mathbb{Q}\langle e_1, \dots, e_5 \rangle$ and $I = (e_1e_2, e_3e_4, (e_1 + e_3)e_5)$ then E/I is not Koszul, since the Betti table of \mathbb{Q} over E/I is

	0	1	2	3	4
total:	1	5	18	57	171
0:	1	5	18	56	160
1:	.	.	.	1	11

(iii) Many results on the exterior algebras have analogues over commutative complete intersections; see, e.g., [1], [2]. However, if $S = k[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ and h a reducible quadratic form, $S/(h)$ is not Koszul in general. For example, take $S = \mathbb{Q}[x, y, z, t]/(x^2, y^2, z^2, t^2)$ and $h = x(y + z + t)$ then $S/(h)$ is not Koszul since the resolution of \mathbb{Q} over $S/(h)$ is

$$\begin{array}{cccccc} & 0 & 1 & 2 & 3 & 4 \\ \text{total:} & 1 & 4 & 11 & 27 & 66 \\ 0: & 1 & 4 & 11 & 25 & 51 \\ 1: & . & . & . & 2 & 15 \end{array}$$

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